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THE STATIC STABILITY PROBLEMS OF AN ELASTIC CONTINUUM SUBJECTED TO FOLLOWER-TYPE LOADING

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Abstract—This study involves the fundamental aspects of stability of elastic systems and its solutions, and presents a new formulation of the static stability criterion. The purpose of the study is to show that by using the constructed self-adjoint boundary problem of a deflected elastic continuum subjected to follower-type loads, a static method of solution may be utilized. To demonstrate this approach we start with the illustrative examples where we can find either a physical analogy for determination of self-adjoint equations, or a mathematical determination of these self-adjoint equations using the Green's integral (for the static solution). Determination of the coefficients for constructing self-adjoint equations can then be accomplished solving the original equations.

NOTATION

$a_{ij}, a'_{ij}, b_{ijk}, b'_{ijk}$	coefficients of applied body and surface traction
A_i, A'_i	coefficients which define a self-adjoint system of differential equations
B	bending rigidity ($B = EJ$)
C	torsional rigidity ($C = GJ_t$)
d, d_1, d_2	distances
E	modulus of elasticity
G	modulus of shear elasticity
J_n	Bessel function
J_t	torsion constant
k	axial load factor for beam column ($k^2 = P/EJ$)
l	length
l_1	distance
m	intensity of mass
n	moment factor ($n^2 = (Px)^2/BC$)
P	axial force in beam column (concentrated force)
P_{cr}	critical buckling load
p_i	components of perturbations of applied surface traction
q	intensity of lateral load
q_i	"lateral" surface loading
Q	concentrated lateral load
Q_i	"lateral" body loading
u, u_i	original displacement
v, v_i	adjoint (self-adjoint) displacement
x_k	rectangular coordinates
X	coefficients ($X = (Pl^2)/2(BC)^{1/2}$)
X_i	components of perturbed applied body traction
$\alpha, \alpha', \beta, \beta'$	parameters associated with the magnitude of coefficients $a_{ij}, a'_{ij}, b_{ijk}, b'_{ijk}$
λ, η	Lamé's constants of elasticity
γ	parameter associated with the magnitude of body and surface tractions
Δ, Δ_i	determinants of Euler critical value of load
$\varepsilon, \varepsilon_i$	expressions for determination of "adjoint" critical value of load P or γ
δ_{ij}	the Kronecker delta
Φ, φ	angle of twist of a bar
Ψ, ψ	angle of twist of a bar of the adjoint problem
σ_{ij}	stress tensor
$\kappa, \kappa_i, \omega_i$	determinants of deflections and their derivatives.

1. INTRODUCTION

Determination of the critical load of conservative problems of elastic stability by the dynamic method is independent of the distribution of mass and determination by the static method is independent of the distribution of "lateral" load. Dynamic and static methods give the same value for the critical load. The critical loads of nonconservative problems are dependent on the distribution of mass in solution by the dynamic method (Bolotin, 1961).

We expect that in the static method of solution, the critical load has to be dependent on the distribution of the "lateral" load.

This assumption may be partially justified by observing that the equations of motion are the same as the equations of equilibrium under certain "lateral" forces—the reversed kinetic reactions.

Using a normal static solution, we obtain an expression for deflection without any indication of a critical load. Using a constructed self-adjoint static method of solution, we can obtain a new form for the expressions of deflection and of critical load together. The expression which defines deflection is identical to the expression we found by normal static consideration. The expression for determination of the critical load gives us the "adjoint" critical load which is dependent on the distribution of the "lateral" load. Note that dynamic and static methods give different answers for the critical loads.

To illustrate this method of solution, let us consider several problems:

(1) Lateral buckling of a cantilever subjected to a transverse follower force.
(2) A cantilever bar subjected at its free end to a load followed by a point on a bar.
(3) A cantilever bar subjected at its free end to a follower force (Beck's problem).
(4) The adjoint expression of the phenomenon of reversal of deflections.

2. LATERAL BUCKLING OF A CANTILEVER SUBJECTED TO A TRANSVERSE FOLLOWER FORCE

As is known, this problem is still unsolved statically. It was, however, solved dynamically by Como (1966). To obtain a solution we split P into vertical $\sim P$ and horizontal $\sim P\varphi(0)$ components (see Fig. 1). The torque is given by:

$$C\varphi' = P(xu' - u). \tag{1}$$

The bending moment is computed for the lateral disributed load q,

$$Bu'' = -Px\varphi + P\varphi(0)x - \frac{qx^2}{2}, \tag{2a}$$

for the lateral concentrated load Q,

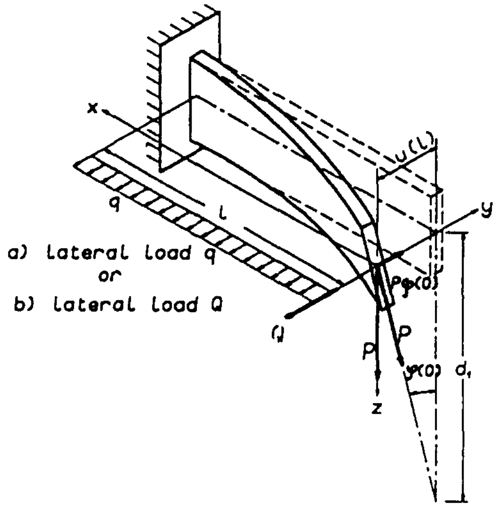

Fig. 1. Lateral buckling of a cantilever subjected to a transverse follower force.

$$Bu'' = -Px\varphi + P\varphi(0)x - Qx. \tag{2b}$$

Differentiating (1) and substituting (2a, b) in the result, we have:

$$\text{for } q, \quad \varphi'' + n^2\varphi - n^2\varphi(0) = -n^2\frac{qx}{2P}, \qquad \text{for } Q, \quad \varphi'' + n^2\varphi - n^2\varphi(0) = -n^2\frac{Q}{P}. \tag{3a, b}$$

where

$$n^2 = \frac{(Px)^2}{BC} \quad \text{and} \quad X = \frac{Pl^2}{2\sqrt{BC}}. \tag{4}$$

The boundary conditions being:

$$(1)\ u(0) = 0; \quad (2)\ \varphi'(0) = 0; \quad (3)\ \varphi(l) = 0; \quad (4)\ u'(l) = 0;$$

$$(5)\ u(l) = -\frac{C}{P}\varphi'(l); \quad (6)\ \varphi(0) = \varphi(0) \tag{5}$$

and the general solutions are

$$\text{for } q, \quad \varphi = A_1\sqrt{x}J_{1/4}\left(X\frac{x^2}{l^2}\right) + A_2\sqrt{x}J_{-1/4}\left(X\frac{x^2}{l^2}\right) + \varphi(0) - \frac{qx}{2P}, \tag{6a}$$

$$\text{for } Q, \quad \varphi = A_3\sqrt{x}J_{1/4}\left(X\frac{x^2}{l^2}\right) + A_4\sqrt{x}J_{-1/4}\left(X\frac{x^2}{l^2}\right) + \varphi(0) - \frac{Q}{P}, \tag{6b}$$

and finally

$$\text{for } q, \quad \varphi(0) = \frac{ql}{2P}\left[1 - 1.0779\frac{J_{1/4}(X)}{X^{1/4}}\right], \qquad \text{for } Q, \quad \varphi(0) = \frac{Q}{P}[1 - 1.0304X^{1/4}J_{-1/4}(X)], \tag{7a, b}$$

$$\text{for } q, \quad u(l) = \frac{C}{Fl^2}\frac{ql}{2P}l[1 - 2.1558X^{3/4}J_{3/4}(X)], \tag{8a}$$

$$\text{for } Q, \quad u(l) = \frac{Q}{P}l\sqrt{\frac{C}{B}}\,1.0304X^{1/4}J_{3/4}(X). \tag{8b}$$

Equations (7a, b) and (8a, b) do not define a critical load P. To do so we determine a distance $d_1 = u(l)/\varphi(0)$ (see Fig. 1). (9)

$$\text{For } q, \quad \frac{d_1}{l}\sqrt{\frac{B}{C}} = \frac{1}{2X}\,\frac{1 - 2.1558X^{3/4}J_{-3/4}(X)}{1 - 1.0779\dfrac{J_{1/4}(X)}{X^{1/4}}}, \quad \text{for } Q, \quad \frac{d_1}{l}\sqrt{\frac{B}{C}} = \frac{1.0304X^{1/4}J_{3/4}(X)}{1 - 1.0304X^{1/4}J_{-1/4}(X)}. \tag{10a, b}$$

Here distance d_1 is dependent on P and type of the lateral load. To illustrate the static method of solution for the determination of critical load, let us compare this problem with the problem of a cantilever subjected to a transverse force P produced by the tension of a

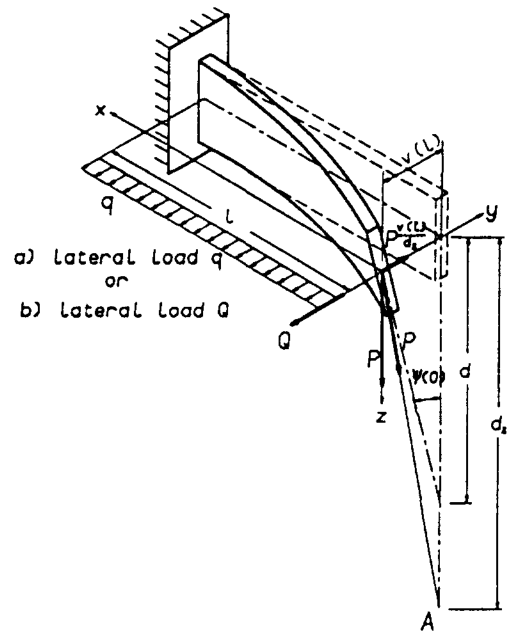

Fig. 2. Lateral buckling of a cantilever subjected to a transverse force produced by the tension of a cable which always passes through the point A.

cable which passes through point A (see Fig. 2). This problem was solved statically by Prandtl (1899).

If we choose distances $d_1 = u(l)/\varphi(0)$ (Fig. 1) and $d_2 = v(l)/\psi(0)$ such that $d_1 = d_2$, we obtain conditions for which the second problem is adjoint:

$$\frac{u(l)}{\varphi(0)} = \frac{v(l)}{\psi(0)} = d. \tag{11}$$

Similarly, we can also obtain expression (11) from Green's integral

$$\int_0^l \{L[\varphi-\varphi(0)][\psi-\psi(0)] - L[\psi-\psi(0)][\varphi-\varphi(0)]\}\,dx$$
$$= [\varphi-\varphi(0)]'[\psi-\psi(0)] - [\varphi-\varphi(0)][\psi-\psi(0)]'|_0^l = u(l)\psi(0) - \varphi(0)v(l) = 0, \tag{12}$$

by substituting the boundary conditions (5) into terms v and ψ.

Here

$$L[\varphi-\varphi(0)] = [\varphi-\varphi(0)]'' + n^2[\varphi-\varphi(0)] = 0 \tag{13}$$

is the left-hand side of eqns (3a, b) and

$$L[\psi-\psi(0)] = [\psi-\psi(0)]'' + n^2[\psi-\psi(0)] = 0 \tag{14}$$

is the self-adjoint differential equation.

The boundary-value problem will be self-adjoint if by virtue of boundary conditions (5) the integral (12) vanishes for any choice of functions u, φ and v, φ satisfying these conditions.

For the case in Fig. 2, the equations of equilibrium of moments can be expressed as follows:

$$P(xv' - v) = C\psi' \tag{15}$$

$$\text{for } q, \quad Bv'' = -Px\psi + P\frac{v(l)}{d_2}x - \frac{qx^2}{2}, \qquad \text{for } Q, \quad Bv'' = Px\psi + P\frac{v(l)}{d_2}x - Qx. \tag{16a, b}$$

The variable v is eliminated from eqns (15), (16a, b) giving the following equations in terms of ψ only

$$\text{for } q, \quad \psi'' + n^2\psi - n^2\frac{v(l)}{d_2} = -n^2\frac{qx}{2P}, \qquad \text{for } Q, \quad \psi'' + n^2 - n^2\frac{v(l)}{d_2} = -n^2\frac{Q}{P}. \tag{17a, b}$$

From eqns (17a, b) we also obtain by substitution $\psi(0) = v(l)[\varphi(0)/u(l)]$ or $\psi(0) = v(l)/d$ from (11) to (3a) and (3b) in terms of ψ. The relevant boundary conditions (5) are in terms of v and ψ. The solutions to these equations, in terms of Bessel functions are:

$$\text{for } q, \quad \psi = A_3\sqrt{x}J_{1/4}\left(X\frac{x^2}{l^2}\right) + A_4\sqrt{x}J_{-1/4}\left(X\frac{x^2}{l^2}\right) + \frac{v(l)}{d_2} - \frac{qx}{2P}, \tag{18a}$$

$$\text{for } Q, \quad \psi = A_5\sqrt{x}J_{1/4}\left(X\frac{x^2}{l^2}\right) + A_6\sqrt{x}J_{-1/4}\left(X\frac{x^2}{l^2}\right) + \frac{v(l)}{d_2} - \frac{Q}{P}. \tag{18b}$$

If we satisfy the boundary conditions, we obtain the characteristic equation for all cases:

$$\Delta = \sqrt{l}\left[J_{-1/4}(X) + \frac{l}{d_2}\sqrt{\frac{C}{B}}J_{3/4}(X)\right]. \tag{19}$$

We rewrite eqn (19) for determination of the critical load

$$\frac{d_2}{l}\sqrt{\frac{B}{C}}J_{-1/4}(X) + J_{3/4}(X) = 0 \tag{20}$$

and deflection

$$\text{for } q, \quad \Delta v(l) = \frac{ql}{P_2}\sqrt{l}\left\{J_{-1/4}(X)[1 - 2.1558X^{3/4}J_{-3/4}(X)] + 2XJ_{3/4}(X)\left[1 - 1.0779\frac{J_{1/4}(X)}{X^{1/4}}\right]\right\} \tag{21a}$$

or after substituting eqn (19), we rewrite eqn (21a):

$$\frac{Pl^{3/2}}{C}\left[J_{-1/4}(X)\frac{d_2}{l}\sqrt{\frac{B}{C}} + J_{3/4}(X)\right]v(l) = \frac{ql}{2P}\sqrt{l}\left\{J_{-1/4}(X)[1 - 2.1558X^{3/4}J_{-3/4}(X)] + 2XJ_{3/4}(X)\left[1 - 1.0779\frac{J_{1/4}(X)}{X^{1/4}}\right]\right\}\frac{d_2}{l}\sqrt{\frac{B}{C}} \tag{22a}$$

and

$$\text{for } Q,\quad \Delta v(l) = \frac{Q}{P} l \sqrt{\frac{C}{B}} \sqrt{l} J_{3/4}(X) \tag{21b}$$

or after substituting eqn (19), we rewrite eqn (21b):

$$\sqrt{l}\left[\frac{d_2}{l}\sqrt{\frac{B}{C}} J_{-1/4}(X) - J_{3/4}(X)\right] v(l) = \frac{Q}{P} l \sqrt{\frac{C}{B}} \sqrt{l} J_{3/4}(X) \frac{d_2}{l}\sqrt{\frac{B}{C}}. \tag{22b}$$

It should be remarked that $v(l)/\psi(0)$ is given by (10a) and (10b). Because $d_1 = d_2 = d$, we can substitute expression (10a) and (10b) into eqns (20), (22a) and (22b).

From the characteristic eqn (20) we obtain the following expression for the critical load:

$$\text{for } q,\quad J_{-1/4}(X)[1 - 2.1558X^{3/4}J_{-3/4}(X)] + 2XJ_{3/4}(X)\left[1 - 1.0779\frac{J_{1/4}(X)}{X^{1/4}}\right] = 0, \tag{23a}$$

and

$$\text{for } Q,\quad J_{3/4}(X) = 0. \tag{23b}$$

Equation (23a) gives us a critical load P for the distributed lateral load q

$$P_{cr1} \cong 6.16(BC)^{1/2}/l^2 \tag{24a}$$

and eqn (23b) gives us a critical load P for the concentrated load Q

$$P_{cr2} \cong 6.984(BC)^{1/2}/l^2. \tag{24b}$$

Making use of the dynamic analysis for concentrated mass at its free end, Como (1966) obtained:

$$P_{cr} \cong 6.99(BC)^{1/2}/l^2. \tag{25}$$

Deflections:

$$\text{for } q,\quad \left\{J_{-1/4}(X)[1 - 2.1558X^{3/4}J_{-3/4}(X)] + J_{3/4}(X)2X\left[1 - 1.0779\frac{J_{1/4}(X)}{X^{1/4}}\right]\right\}$$
$$\times \left\{v(l) - \frac{ql}{2P} l \frac{C}{Pl^2}[1 - 2.1558X^{3/4}J_{-3/4}(X)]\right\} = 0 \tag{26a}$$

and

$$\text{for } Q,\quad J_{3/4}(X)\left[v(l) - \frac{Q}{P} l \sqrt{\frac{C}{B}}\, 1.0304X^{1/4}J_{3/4}(X)\right] = 0. \tag{26b}$$

These equations are expanded into two expressions. In discussing the solutions of eqns (26a, b), let us begin with two cases:

(a) As long as the first expression is not equal to zero, the second expression will be equal to zero and will give a deflection identical to the one found by normal static consideration [see eqns (8a, b)].

(b) If the first expression is equal to zero we can determine the critical load [see eqns (23a, b)]. Now, because the second expression cannot be zero, the deflection is unstable.

In conclusion, by applying the adjoint concept of the solution for the problem with a follower type load, we can determine the critical load by solving the characteristic equations of the self-adjoint problem. This permits a new form of the expression of the deflection and critical load together, which we were unable to find by normal static consideration.

The original form of equilibrium is stable only up to a certain force, P. At a force exceeding this value, which will be termed critical as before, a transition takes place to a certain new form of instability with ever increasing deflection from the original equilibrium position.

The stability criterion is the condition for the occurrence of the above form of instability and is called the adjoint stability criterion.

3. A CANTILEVER BAR SUBJECTED AT ITS FREE END TO LOAD FOLLOWED BY A POINT ON A BAR

To further illustrate the adjoint method of solution, let us consider a cantilever bar subjected at its free end to a load followed by point B on a bar (see Fig. 3).

We divide P into vertical $\sim P$ and horizontal $\sim P[u(l_1)/l_1]$ components.

The differential equation for this case is:

with uniform load q,

$$EJu''+Pu-P\frac{u(l_1)}{l_1}x=-\frac{qx^2}{2} \tag{27a}$$

and with concentrated load Q,

$$EJu''+Pu-P\frac{u(l_1)}{l_1}x=-Qx. \tag{27b}$$

The solutions to these equations are

$$\text{for } q,\quad u=A_1\sin kx+A_2\cos kx+\frac{u(l_1)}{l_1}x-\frac{qx^2}{2P}+\frac{q}{k^2P} \tag{28a}$$

and

$$\text{for } Q,\quad u=A_3\sin kx+A_4\cos kx+\frac{u(l_1)}{l_1}x-\frac{Q}{P}x \quad \text{where } k^2=P/EJ \tag{28b}$$

which must satisfy the conditions:

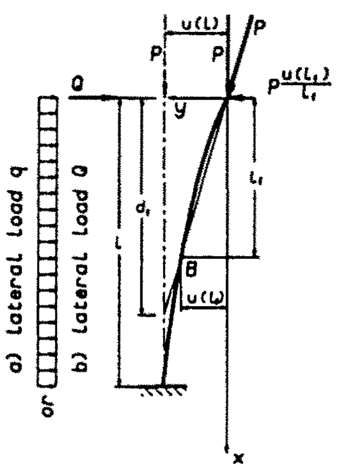

Fig. 3. The cantilever bar subjected at its free end to load followed by point B on a bar.

$$u(0) = 0; \quad u'(l) = 0; \quad u(l_1) = u(l_1) \quad \text{and} \quad u(l) = u(l) \tag{29}$$

for the next case.

From (28a) and (29) we obtain for q:

$$u(l_1) = \frac{ql^2}{P}\left[\left(1 - \frac{\sin kl}{kl}\right)\sin kl_1 - (l_1/l)^2\left(\frac{\cos kl_1 - 1}{k^2 l_1^2} + \frac{1}{2}\right) kl \cos kl\right] \Big/ \frac{l}{l_1}\sin kl_1 \tag{30a}$$

$$u(l) = \frac{ql^2}{P}\left[\frac{l_1}{l}\left(\frac{\cos kl_1 - 1}{k^2 l_1^2} + \frac{1}{2}\right)(\sin kl - kl\cos kl) + \frac{l}{l_1}\sin kl_1\left(\frac{1}{2} - \frac{\cos kl}{k^2 l^2} - \frac{\sin kl}{kl} + \frac{1}{k^2 l^2}\right)\right] \Big/ \frac{l}{l_1}\sin kl_1 \tag{31a}$$

and from (28b) and (29) we obtain for Q:

$$u(l_1) = \frac{Q}{P} l\left(\sin kl_1 - \frac{l_1}{l} kl \cos kl\right) \Big/ \frac{l}{l_1}\sin kl_1 \tag{30b}$$

$$u(l) = \frac{Q}{P} l(\sin kl - kl\cos kl) \Big/ \frac{l}{l_1}\sin kl_1 \tag{31b}$$

or generally, for any distribution of lateral load q:

$$u(l) = \kappa(q, kl)/\Delta_1(kl). \tag{31c}$$

The characteristic equation for these cases is:

$$\frac{l}{l_1}\sin kl_1 = 0 \quad \text{and} \quad P_{cr} = \pi^2 EJ/l_1^2. \tag{32, 33}$$

For a small value l_1 we obtain a large value of critical load P_{cr}. This is impossible.

Let us compare this problem with the problem of a bar with a load P through a fixed point A (see Fig. 4).

A bar loaded by P through a fixed point A, was solved statically by Feodosyev (1950, 1970).

According to Figs 3 and 4 for $d_1 = d_2 = d$ we obtain

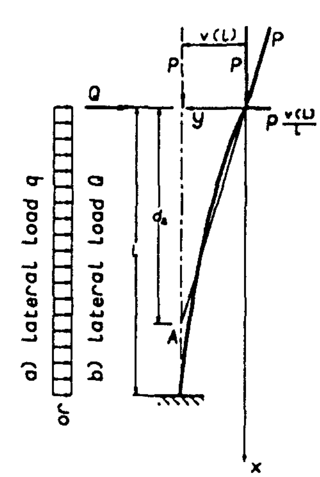

Fig. 4. The cantilever bar with load P through a fixed point A.

$$u(l)/u(l_1) = v(l)/v(l_1) = d/l_1. \tag{34}$$

After differentiation of the left-hand eqns (27a, b) twice, we obtain a self-adjoint form of the differential equations:

$$L(u) = EIu'''' + Pu'' \quad \text{and} \quad L(v) = EJv'''' + Pv''. \tag{35, 36}$$

We can also obtain expression (34) from Green's integral, using boundary conditions (29) in terms of v:

$$\int_0^l [vL(u) - uL(v)]\,\mathrm{d}x = \frac{P}{l_1}[v(l)u(l_1) - u(l)v(l_1)] = 0. \tag{37}$$

We define expression (34) by substitution of (30a) and (31a) for q and (30b), (31b) for Q, respectively:

$$\frac{d}{l} = \frac{\left(\frac{l_1}{l}\right)^2\left(\frac{\cos kl_1 - 1}{k^2l^2} + \frac{1}{2}\right)(\sin kl - kl\cos kl) + \sin kl_1\left(\frac{1}{2} - \frac{\cos kl - 1}{k^2l^2} - \frac{\sin kl}{kl}\right)}{\left(1 - \frac{\sin kl}{kl}\right)\sin kl_1 - \left(\frac{l_1}{l}\right)^2\left(\frac{\cos kl_1 - 1}{k^2l^2} + \frac{1}{2}\right)kl\cos kl}, \tag{38a}$$

$$\frac{d}{l} = \frac{\sin kl - kl\cos kl}{\frac{l}{l_1}\sin kl_1 - kl\cos kl}. \tag{38b}$$

Dividing the force P into vertical and horizontal components (see Fig. 4):

$$P_v \cong P, \qquad P_h \cong \frac{v(l)}{d_2}, \tag{39, 40}$$

and rewriting the equations of the elastic curve of the bar:

$$\text{for } q,\quad EJv'' + Pv - P\frac{v(l)}{d_2}x = -\frac{qx^2}{2}, \qquad \text{for } Q,\quad EJv'' + Pv - P\frac{v(l)}{d_2}x = -Qx. \tag{41a, b}$$

We can obtain adjoint equations (41a, b) by substituting $v(l_1) = v(l)(l_1/d)$ from (34) into (37a) and (27b) for $d_1 = d_2 = d$ in terms of v.

The conditions (29) also in terms of v. The solutions to these equations are:

$$\text{for } q,\quad v = A_5\sin kx + A_6\cos kx + \frac{v(l)}{d_2} - \frac{qx^2}{2P} + \frac{q}{k^2P}, \tag{42a}$$

$$\text{for } Q,\quad v = A_7\sin kx + A_8\cos kx + \frac{v(l)}{d_2} - \frac{Q}{P}x. \tag{42b}$$

Satisfying the boundary conditions (29), we obtain the characteristic equation for all cases of lateral load

$$\Delta = \frac{l}{d_2}(kl\cos kl - \sin kl) - kl\cos kl \tag{43}$$

or for determination of critical load:

$$kl\cos kl - \sin kl - \frac{d_2}{l}kl\cos kl = 0 \tag{44}$$

and deflection for q:

$$\Delta v(l) = \frac{ql^2}{P}\left(\frac{kl\cos kl}{2} - \sin kl + \frac{1}{kl} - \frac{\cos kl}{kl}\right). \tag{45a}$$

We rewrite eqn (45a) after substitution of (43)

$$\left[(kl\cos kl - \sin kl) - \frac{d_2}{l}kl\cos kl\right]v(l) = \frac{ql^2}{P}\left(\frac{kl\cos kl}{2} - \sin kl + \frac{1}{kl} - \frac{\cos kl}{kl}\right)\frac{d_2}{l}. \tag{46a}$$

And for Q,

$$\Delta v(l) = \frac{Q}{P}l(kl\cos kl - \sin kl). \tag{45b}$$

We rewrite eqn (45b) after substituion of (43)

$$\left(kl\cos kl - \sin kl - \frac{d_2}{l}kl\cos kl\right)v(l) = \frac{Q}{P}l\left(kl\cos kl - \sin kl\right)\frac{d_2}{l}. \tag{46b}$$

Because $d_1 = d_2 = d$, we can substitute expressions (38a, b) into (44). Now, from the characteristic eqn (44) we obtain expressions for the determination of the critical load:

$$\text{for } q,\quad \frac{l}{l_1}\sin kl_1\left(\frac{kl\cos kl}{2} - \sin kl + \frac{1}{kl} - \frac{\cos kl}{kl}\right) = 0, \tag{47a}$$

$$\text{for } Q,\quad \frac{l}{l_1}\sin kl_1(kl\cos kl - \sin kl) = 0, \tag{47b}$$

or generally, the adjoint characteristic equation is:

$$\Delta_1(kl)\varepsilon(kl) = 0. \tag{47c}$$

Equations (47a, b, c) are expanded into two expressions. From the first expression of eqns (47a, b), ($\sin kl_1 = 0$) and of eqn (47c), [$\varepsilon(kl) = 0$] we obtain a critical load exactly like we found by normal static consideration [see eqn (32)]. The second expression of eqn (47a) gives us for a distributed load q

$$\frac{kl\cos kl}{2} - \sin kl + \frac{1}{kl} - \frac{\cos kl}{kl} = 0 \tag{48a}$$

and an "adjoint" critical load equal to:

$$P_{cr} \cong 16.69EJ/l^2. \tag{49a}$$

The second expression of eqn (47b) gives us, for concentrated load Q,

$$\frac{\tan kl}{kl} = 1, \tag{48b}$$

and an "adjoint" critical load equal to

$$P_{cr} \cong 20.19EJ/l^2, \tag{49b}$$

and the second expression of eqn (47c) for any distribution of q

$$\varepsilon(kl) = 0, \tag{48c}$$

which gives us a generally "adjoint" critical load. Using the adjoint expression of deflection with distributed load q, we obtain for $d_1 = d_2 = d$ and after substitution of eqn (38a) into (46a):

$$\left(\frac{kl\cos kl}{2} - \sin kl + \frac{1}{kl} - \frac{\cos kl}{kl}\right)\left\{v(l) - \frac{ql^2}{P}\left[\frac{l_1}{l}\left(\frac{\cos kl_1 - 1}{k^2 l_1^2} + \frac{1}{2}\right)(\sin kl - kl\cos kl)\right.\right.$$
$$\left.\left. + \frac{l}{l_1}\sin kl_1\left(\frac{1}{2} - \frac{\cos kl - 1}{k^2 l^2} - \frac{\sin kl}{kl}\right)\right]\Big/\frac{l}{l_1}\sin kl_1\right\} = 0, \tag{50a}$$

and an adjoint expression of deflection with concentrated load Q for $d_1 = d_2 = d$ and after substitution of eqn (38b) into (46b):

$$(kl\cos kl - \sin kl)\left[v(l) - \frac{Q}{P}l\frac{\sin kl - kl\cos kl}{\frac{l}{l_1}\sin kl_1}\right] = 0 \tag{50b}$$

and generally

$$\varepsilon(kl)\left[v(l) - \frac{\kappa(q, kl)}{\Delta_1(kl)}\right] = 0. \tag{50c}$$

Equations (50a, b, c) are expanded into two expressions. The first expression defines the "adjoint" critical load [see eqns (48a, b, c)] and the second expression defines deflection, exactly as we found it by normal static consideration [see eqns (31a, b, c)]. It should be noted that from eqn (50c), we can obtain another equivalent form:

$$\varepsilon(kl)v(l) - \frac{\kappa(q, kl)\varepsilon(kl)}{\Delta_1(kl)} = 0 \tag{51}$$

or finally

$$v(l) = \frac{\kappa(q, kl)\varepsilon(kl)}{\Delta_1(kl)\varepsilon(kl)}. \tag{52}$$

If we cancel out a factor $\varepsilon(kl)$ from the numerator and denominator, we lose the "adjoint" solution. For example, if we write eqn (45a) in the form

$$v(l) = \frac{ql^2}{P}\left(\frac{kl\cos kl}{2} - \sin kl + \frac{1}{kl} - \frac{\cos kl}{kl}\right)\Big/\Delta \tag{53}$$

after substituting eqn (38a) we obtain:

$$v(l) = \frac{ql^2}{P}\left[\frac{l_1}{l}\left(\frac{\cos kl_1 - 1}{k^2l^2} + \frac{1}{2}\right)(\sin kl - kl\cos kl) + \frac{l}{l_1}\sin kl_1\left(\frac{1}{2} - \frac{\cos kl - 1}{k^2l^2} - \frac{\sin kl}{kl}\right)\right]\left(\frac{kl\cos kl}{2} - \sin kl - \frac{\cos kl - 1}{kl}\right)\Big/ \frac{l}{l_1}\sin kl_1\left(\frac{kl\cos kl}{2} - \sin kl - \frac{\cos kl - 1}{kl}\right). \tag{54}$$

This means that if we obtain the equivalent form (52) and go to the "adjoint" form (50c) instead of canceling out, we will obtain an adjoint solution.

4. A CANTILEVER BAR SUBJECTED AT ITS FREE END TO FOLLOWER FORCE [BECK'S PROBLEM (1952)]

The solution to this problem can be obtained from the previous case of a cantilever bar subjected at its free end to a load followed by a point on a bar, when the point on a bar approaches its free end ($l_1 \to 0$, see Figs 3 and 5).

From characteristic eqns (47a, b) we obtain:

$$\text{for } q,\quad kl\left(\frac{kl\cos kl}{2} - \sin kl - \frac{\cos kl - 1}{kl}\right) = 0, \tag{55a}$$

$$\text{for } Q,\quad kl(kl\cos kl - \sin kl) = 0. \tag{55b}$$

From eqns (52a, b),

$$\text{for } q,\quad \left(\frac{kl\cos kl}{2} - \sin kl - \frac{\cos kl - 1}{kl}\right)\left[v(l) - \frac{ql^2}{P}\left(\frac{1}{2} + \frac{1-\cos kl}{k^2l^2} - \frac{\sin kl}{kl}\right)\right] = 0 \tag{56a}$$

and the adjoint expression of deflection with concentrated load Q is,

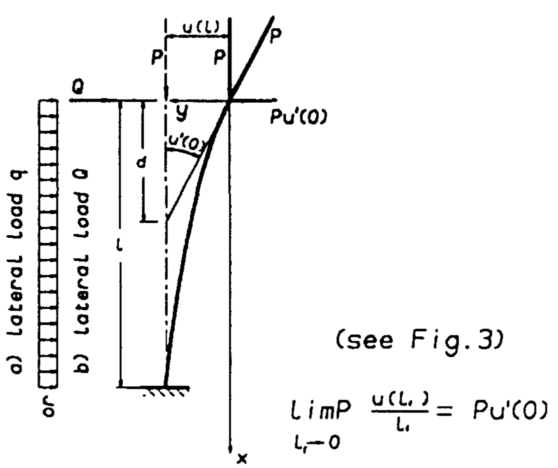

Fig. 5. The cantilever bar subjected at its free end to follower force (Beck's problem).

$$(kl\cos kl-\sin kl)\left[v(l)-\frac{Q}{P}l\left(\cos kl-\frac{\sin kl}{kl}\right)\right]=0. \tag{56b}$$

It is a full static solution of the stability of the deflected elastic bar subjected to a follower type load.

Let us examine the solutions to eqns (56a, b). These eqns are expanded into two expressions. The first expression defines the "adjoint" critical load [see also (48a), (49a) and (48b), (49b)].

Making use of the dynamic analysis for distributed mass, Beck (1952) obtained

$$P_{cr}\cong 20.05EJ/l^2. \tag{57}$$

The "adjoint" critical load is [see also (55a) and solution (49a)]:

$$P_{cr}\cong 16.69EJ/l^2. \tag{58}$$

Making use of the dynamic analysis for concentrated mass M, Dzhenalidze (1958) obtained

$$P_{cr}=20.19EJ/l^2. \tag{59}$$

The "adjoint" critical load, for concentrated load Q is the same [see (49b)].

The second expression in eqns (56a, b) gives us deflection, exactly like that which is found by normal static consideration.

5. THE ADJOINT EXPRESSION OF THE PHENOMENON OF REVERSAL OF DEFLECTIONS

It is interesting to note that the deflections produced by the two opposite eccentrically applied compressive forces P on a beam of length l on two simple supports may reverse direction during a continuous increase in the value of P. It is a purely conservative problem.

The differential equation of the elastic curve for this case (see Fig. 6) is:

$$v''+k^2v=k^2e\left(\frac{2x}{l}-1\right). \tag{60}$$

The solution of eqn (60) according to the conditions at the ends takes the form

$$v=e\left(\cos kx-\frac{1+\cos kl}{\sin kl}\sin kx+2x/(l-1)\right) \tag{61}$$

or in an equivalent form

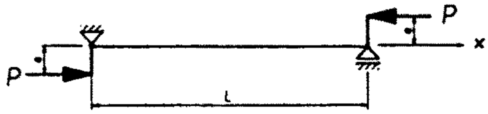

Fig. 6. The simple supported beam subjected to the opposite eccentrically compressive force P.

$$v = e\left[\cos kx - \frac{\cos^2 \frac{kl}{2}}{\sin \frac{kl}{2} \cos \frac{kl}{2}} \sin kx + 2x/(l-1)\right], \tag{62}$$

where e is an eccentric and $k^2 = P/EJ$. By multiplying the solution (62) by $\cos(kl/2)$ and making corresponding changes, we obtain a final adjoint expression of the deflection:

$$\cos \frac{kl}{2} \left\{v - e\left[\cos kx - \sin kx/\tan \frac{kl}{2} + 2x/(l-1)\right]\right\} = 0. \tag{63}$$

There are two possibilities for the left-hand side of (63) to become zero. Either $\cos(kl/2) = 0$, which gives for the adjoint critical load the value

$$P_{cr} = \pi^2 EJ/l^2, \tag{64}$$

or the expression in the brackets may become zero and define the well known conventional deflection which does not indicate the real critical load:

$$P_{cr} = 4\pi^2 EJ/l^2. \tag{65}$$

This solution of the phenomenon of reversal of deflections makes the concept of the adjoint critical load of nonconservative problems more understandable.

6. THE STATIC STABILITY OF AN ELASTIC CONTINUUM

Let us consider an isotropic, homogeneous, elastic solid occupying a volume V bounded by a finite surface S. It will be assumed that on one part of the boundary of the solid S_0 the displacements are precribed so as to preclude a rigid body motion. Referring to an orthogonal Cartesian coordinate system x_j (Novozhilov, 1948; Boloton, 1961) has obtained the following equations for the static boundary-value problem:

$$\frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta}\right) + \gamma \frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_k}\right) + \gamma X_i = 0, \quad \text{in } V, \tag{66}$$

with the following boundary conditions on the surface:

$$\lambda_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} n_j + \gamma\sigma_{ij}\frac{\partial u_i}{\partial x_k} n_j - \gamma p_i = 0, \quad \text{on } S - S_0, \qquad u_i = 0, \quad \text{on } S_0, \tag{67, 68}$$

where

$$\lambda_{ij\alpha\beta} = \lambda\delta_{ij}\delta_{\alpha\beta} + 2\mu\delta_{i\alpha}\delta_{j\beta}, \quad \text{and} \quad \delta_{ij} = 0, 1 \quad i \neq j, i = j. \tag{69, 70}$$

In eqns (66), (67) u_i is the displacement vector measured from the undisturbed state and n_j is the outward positive unit normal vector to S, γ is a parameter associated with the magnitude of body and surface tractions. In eqn (69), λ and μ are Lamé's constants of elasticity. The repeated indices are summed over the range of their definition and X_i, p_i are the components of the perturbations of the applied body and surface traction and their form will depend on the behavior of the nonconservative forces. They will generally be homogeneous functions of displacement and their derivatives with respect to space. For this present study, however, it suffices to restrict X_i and p_i to the following expressions:

$$X_i = \alpha a_{ij} u_j + \beta b_{ijk} \frac{\partial u_j}{\partial x_k}, \qquad p_i = \alpha' a'_{ij} u_j + \beta' b'_{ijk} \frac{\partial u_j}{\partial x_k}, \tag{71, 72}$$

where a_{ij}, b_{ijk}, a'_{ij}, b'_{ijk} are coefficients which are independent of the vector u_i and its derivatives but in general are functions of spatial coordinates x_j, and $\alpha, \beta, \alpha', \beta'$ are parameters associated with magnitude of these coefficients.

The solutions of the homogeneous system of eqns (66) with the boundary conditions (67) with (71), (72) give the characteristic equations:

$$\Delta_i(\alpha, \beta, \alpha', \beta', \gamma) = 0, \tag{73}$$

and eqn (73) sometimes gives, for some range of coefficients, $\alpha, \beta, \alpha', \beta'$, the critical value of γ.

If a body and a surface are subjected to the additional "lateral" body loading Q_i and surface loading q_i, by solving nonhomogeneous system of equations, we obtain expressions for the deflections

$$u_i = \frac{\kappa_i(x_k, \gamma, Q_i, q_i)}{\Delta_i(\alpha, \beta, \alpha', \beta', \gamma)} \tag{74}$$

and their derivatives

$$\frac{\partial u_i}{\partial x_k} = \frac{\omega_i(x_k, \gamma, Q_i, q_i)}{\Delta_i(\alpha, \beta, \alpha', \beta', \gamma)}. \tag{75}$$

The boundary value problem is self-adjoint if by virtue of the boundary conditions, the expression

$$\int_V v_i L_i(u_1, u_2, u_3)\,\mathrm{d}V - \int_V u_i L_i(v_1, v_2, v_3)\,\mathrm{d}V \tag{76}$$

vanishes for any choice of vectors u_1, u_2, u_3 and v_1, v_2, v_3 satisfying these conditions. Here L_i are the left-hand side of eqns (66).

After integrating we have (Bolotin, 1961):

$$\int_V [v_i X_i(u_1, u_2, u_3) - u_i X_i(v_1, v_2, v_3)]\,\mathrm{d}V + \iint_S [v_i p_i(u_1, u_2, u_3) - u_i p_i(v_1, v_2, v_3)]\,\mathrm{d}S = 0. \tag{77}$$

After substituting (71) and (72) into (77) we obtain:

$$\int_V \left[a_{ij}(v_i u_j - u_i v_j) + b_{ijk}\left(v_i \frac{\partial u_j}{\partial x_k} - u_i \frac{\partial v_j}{\partial x_k} \right) \right] \mathrm{d}V + \iint_S \left[a'_{ij}(v_i u_j - u_i v_j) + b'_{ijk}\left(v_i \frac{\partial u_i}{\partial x_k} - u_i \frac{\partial v_j}{\partial x_k} \right) \right] \mathrm{d}S = 0. \tag{78}$$

From the well known conditions $u_k = 0$, $\partial u_j/\partial x_k = 0$ and their combinations which can fill eqn (78), we have other conditions:

$$\frac{v_j}{v_i} = \frac{u_j}{u_i} = \zeta_{ij}, \text{ under the body forces,} \qquad \frac{v_j}{v_i} = \frac{u_j}{u_i} = \zeta'_{ij}, \text{ under the surface forces,}$$

$$\frac{\dfrac{\partial v_j}{\partial x_k}}{v_i} = \frac{\dfrac{\partial u_j}{\partial x_k}}{u_i} = \vartheta_{ijk}, \text{ under the body forces,} \qquad \frac{\dfrac{\partial v_j}{\partial x_k}}{v_i} = \frac{\dfrac{\partial u_j}{\partial x_k}}{u_i} = \vartheta'_{ijk}, \text{ under the surface forces.} \tag{79}$$

Now we can determine the relations (79) from the expressions (74) and (75) and they are dependent on the disribution of "lateral" loads Q_i and q_i. These determined relations (79) provide the possibility to construct a self-adjoint system of equations, which fulfills conditions (76), (77).

$$\frac{\partial}{\partial x_i}\left(\lambda_{ij\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta}\right)+\gamma\frac{\partial}{\partial x_i}\left(\sigma_{jk}\frac{\partial v_i}{\partial x_k}\right)+\gamma A_i v_i = 0, \qquad \lambda_{ij\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta}n_j+\gamma\sigma_{ij}\frac{\partial v_i}{\partial x_k}n_j-\gamma A'_i v_i = 0, \tag{80, 81}$$

where

$$A_i = \alpha a_{ij}\zeta_{ij}+\beta b_{ijk}\vartheta_{ijk}, \qquad A'_i = \alpha' a'_{ij}\zeta'_{ij}+\beta' b'_{ijk}\vartheta'_{ijk}. \tag{82, 83}$$

Now, because these equations are self-adjoint, we can find the critical value of γ. The expected solutions of this homogeneous system of eqns (80) with boundary conditions (81) are characteristic equations [see eqns (23a, b), (47a, b, c), (55a, b)]:

$$\Delta_i(\alpha,\beta,\alpha',\beta',\gamma)\varepsilon_i(\gamma) = 0. \tag{84}$$

Equations (84) are expanded into two expressions. From the first expression

$$\Delta_i(\alpha,\beta,\alpha',\beta',\gamma) = 0, \tag{85}$$

we obtain the critical value of γ for some range of coefficients $\alpha, \beta, \alpha', \beta'$, as we found by normal static consideration [see eqn (73)]. The second expression of eqns (84) gives us the "adjoint" critical value of γ. If a body and a surface are subjected to the same additional "lateral" body load Q_i and surface load q_i, the expected solution of deflections are [see eqns (26a, b), (50a, b, c)]:

$$\varepsilon_i(\gamma)\left[v_i-\frac{\kappa_i(x_k,\gamma,Q_i,q_i)}{\Delta_i(\alpha,\beta,\alpha',\beta',\gamma)}\right] = 0, \tag{86}$$

or in the equivalent form [see (52) and (54)]:

$$v_i = \frac{\kappa_i(x_k,\gamma,Q_i,q_i)}{\Delta_i(\alpha,\beta,\alpha',\beta',\gamma)}\frac{\varepsilon_i(\gamma)}{\varepsilon_i(\gamma)}. \tag{87}$$

It is the expected full static solution of deflected elastic continua subjected to follower type loads.

Equations (86) are expanded into two expressions. The first expression gives an "adjoint" critical value of γ, and the second gives a deflection such as was found by normal static consideration [see eqn (74)].

We can remark that, eqns (66) with (71) and boundary conditions (67) with (72) are identical to eqns (80) and boundary conditions (81), which may be proved as follows.

Equations (80) can be written as:

$$\frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta}\right)+\gamma\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial v_i}{\partial x_k}\right)+\gamma A_i v_i-\gamma\alpha a_{ij}v_j-\gamma\beta b_{ijk}\frac{\partial v_j}{\partial x_k}$$

$$+\gamma\alpha a_{ij}v_j-\gamma\beta b_{ijk}\frac{\partial v_j}{\partial x_k}=0; \quad (88)$$

we observe that

$$\gamma A_i v_i-\gamma\alpha a_{ij}v_j-\gamma\beta b_{ijk}\frac{\partial v_j}{\partial x_k}=0, \quad (89)$$

after substitution.

$$\zeta_{ij}=\frac{v_j}{v_i} \quad \text{and} \quad \vartheta_{ijk}=\frac{\dfrac{\partial v_j}{\partial x_k}}{v_i}, \quad (90)$$

from (79).

Now we obtain eqns (66) and (71) in terms of v. Boundary conditions (81) can be written as:

$$\lambda_{ij\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta}n_j+\gamma\sigma_{jk}\frac{\partial v_i}{\partial x_k}n_j-\gamma A_i' v_i+\gamma\alpha' a_{ij}' v_j+\gamma\beta' b_{ijk}'\frac{\partial v_j}{\partial x_k}-\gamma\alpha' a_{ij}' v_j+\gamma\beta' b_{ijk}'\frac{\partial v_j}{\partial x_k}=0. \quad (91)$$

We observe that

$$-\gamma A_i' v_i+\gamma\left(\alpha' a_{ij}' v_j+\beta' b_{ijk}'\frac{\partial v_j}{\partial x_k}\right)=0, \quad (92)$$

after substitution,

$$\zeta_{ij}'=\frac{v_j}{v_i} \quad \text{and} \quad \vartheta_{ijk}'=\frac{\dfrac{\partial v_j}{\partial x_k}}{v_i}, \quad (93)$$

from (79) to A'.

Now boundary conditions (91) have the form of (67) with (72) in terms of v. We must remark again that eqns (80) with boundary conditions (81) are self-adjoint, because they are constructed by conditions (76), (77).

7. CONCLUSIONS

(1) Both sets of differential equations and boundary conditions give the same expression for deflection.

(2) Only a self-adjoint set of differential equations and boundary conditions can give an expression for the critical load.

(3) The adjoint method complements the static method of solution (divergence).

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